Analysis IV

(for EL, GM, MX)

Week 12 - 12.05.25

Applications to partial differential equations

We will apply the theory of Fourier / Laplace transforms to solve initial - boundary value problems for linear partial differential equations with constant coefficients.

E.g. : heat equation :

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & \text{for } t>0 \text{ and } 0<x<L \\ u(x,0) = u_0(x) & \text{(Initial condition)} \\ u(0,t) = 0 \,,\ u(L,t) = 0 & \text{(boundary cond.)} \end{cases}$$

Describes the evolution of the temperature of a 1-dimensional rod (interval $0 < x < L$) with heat source f and fixed temperature at the endpoints $x = 0, L$.

General procedure that we will follow for such problems:

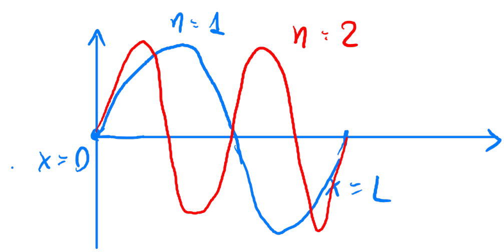
- Fourier series decomposition in directions corresponding to bounded intervals ($x \in (0, L)$)
- Fourier transform in doubly infinite directions ($x \in (-\infty, +\infty)$)
- Laplace transform for semi-infinite directions ($t \in [0, +\infty)$)

Recap on Fourier series

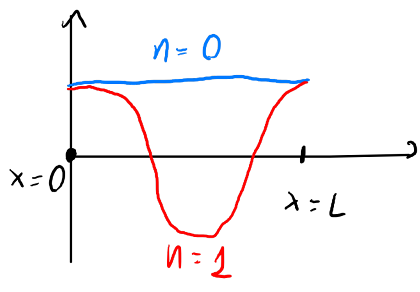
We want to decompose functions $f : [0, L] \longrightarrow \mathbb{R}$ into an infinite sum of simple oscillating functions.

Basis functions:

- $\sin\left(\frac{2\pi n}{L} x\right)$, $n \geq 1$



- $\cos\left(\frac{2\pi n}{L} x\right)$, $n \geq 0$



- Orthogonality relations:

If we denote $\langle f, g \rangle := \int_0^L f(x)\, g(x)\, dx$

(this is indeed an "inner-product" on the space of functions on $[0, L]$)

- $\left\langle \sin\left(\frac{2\pi n}{L} x\right), \sin\left(\frac{2\pi m}{L} x\right) \right\rangle = \begin{cases} L/2, & n = m \\ 0, & n \neq m \end{cases}$

- Same for $\cos\left(\frac{2\pi n}{L} x\right), \cos\left(\frac{2\pi m x}{L}\right)$

- $\left\langle \sin\left(\frac{2\pi n}{L} x\right), \cos\left(\frac{2\pi m}{L} x\right) \right\rangle = 0$ for all n, m

We use these functions as an orthogonal basis to decompose other functions.

Trigonometric (Fourier) series:

Given a function $f: [0,L] \longrightarrow \mathbb{R}$ which is piecewise continuous, can we decompose

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\left(\frac{2\pi n}{L} x\right) + b_n \cdot \sin\left(\frac{2\pi n}{L} x\right) \right)$$

?

(The same considerations in the case when $f: \mathbb{R} \to \mathbb{R}$ is periodic with period L; I can think of it as a function on $[0,L]$)

The answer is yes!

Note that if such a decomposition exists: I can compute (formally) because of the orthogonality conditions:

- $\displaystyle\int_0^L f(x)\,dx = \frac{L}{2} a_0$

- $\displaystyle\int_0^L f(x) \cos\left(\frac{2\pi n}{L} x\right) dx = \frac{L}{2} a_n \quad , n = 1, 2, \ldots$

- $\displaystyle\int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) dx = \frac{L}{2} b_n, \quad n = 1, 2, \ldots$

So it makes sense to define

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{2\pi n}{L} x\right) dx, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) dx, \quad n = 0, 1, 2, \ldots$$

Then: If we denote by $f_N(x)$

the partial sum

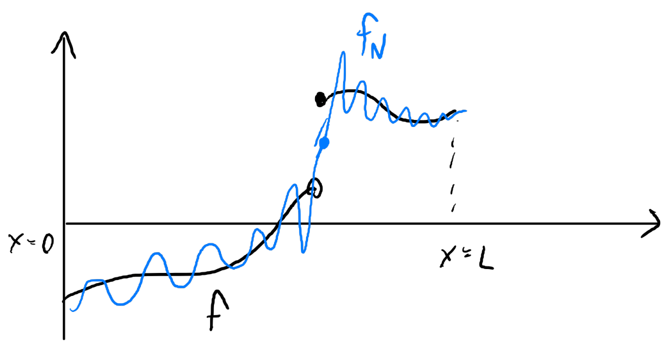
$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cdot \cos\left(\frac{2\pi n}{L} x\right) + b_n \cdot \sin\left(\frac{2\pi n}{L} x\right)\right)$$

with a_n, b_n defined as above, the following theorem holds:

Theorem: Assume that $f : [0, L] \to \mathbb{R}$ is piecewise continuous. Then

$f_N(x) \xrightarrow{N \to +\infty} f(x)$ at every point x where f is differentiable. Moreover

$$\int_0^L |f_N(x) - f(x)|\, dx \xrightarrow{N \to +\infty} 0.$$



At points of jump discontinuities: $f_N \longrightarrow \frac{f_+ + f_-}{2}$ (midpoint).

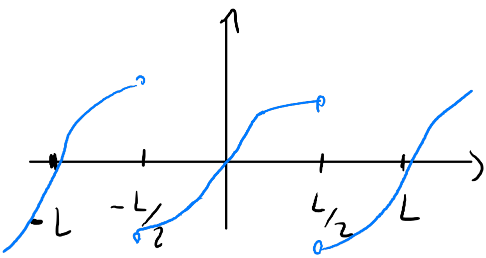
As we said, Fourier series decomposition:
Also for functions $f: \mathbb{R} \to \mathbb{R}$ which are L-periodic.

Special cases:

- When f is odd $(f(-x) = -f(x))$:

Then $a_n = 0$ and $f(x) = \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{2\pi n}{L} x\right)$

With $b_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) dx$



(the integral is the same over any interval of length L since both f and $\sin\left(\frac{2\pi n}{L} x\right)$ are L periodic)

$$= \frac{2}{L} \int_{-L/2}^{+L/2} f(x) \cdot \sin\left(\frac{2\pi n}{L} x\right) dx$$

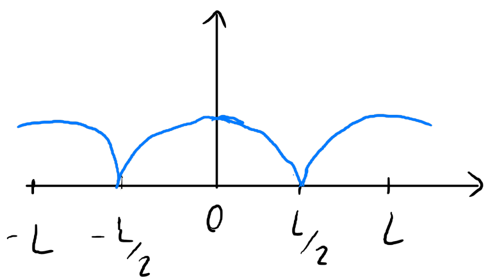
(Here: use the fact that $f(-x) = -f(x)$ and $\sin\left(\frac{2\pi n}{L}(-x)\right) = -\sin\left(\frac{2\pi n}{L} x\right)$)

$$= \frac{4}{L} \int_0^{L/2} f(x) \sin\left(\frac{2\pi n}{L} x\right) dx$$

- If f is even $\left(f(-x) = f(x)\right)$

  Then $b_n = 0$, so

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{2\pi n}{L} x\right)$$



with

$$a_n = \frac{2}{L} \int_0^L f(x) \cdot \cos\left(\frac{2\pi n}{L} x\right)$$

$$= \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cdot \cos\left(\frac{2\pi n}{L} x\right)$$

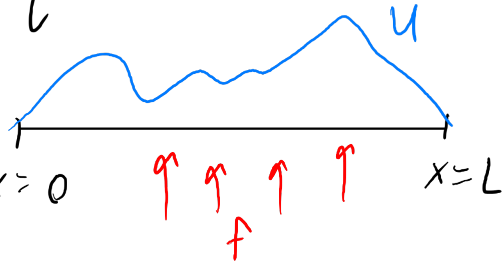
$$= \frac{4}{L} \int_0^{L/2} f(x) \cos\left(\frac{2\pi n}{L} x\right).$$

Remarks:

- The functions $a_n \cdot \cos\left(\frac{2\pi n}{L} x\right)$, $b_n \cdot \sin\left(\frac{2\pi n}{L} x\right)$ are sometimes called **modes**.
- $e_n = \cos\left(\frac{2\pi n}{L} x\right)$: $\quad e_n'(0) = 0 = e_n'(L)$

- $h_n = \sin\left(\frac{2\pi n}{L} x\right)$: $\quad h_n(0) = 0 = h_n(L)$.

Let's return to the 1-dimensional heat equation:



$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f & \text{for } t>0, \quad 0<x<L \\ u\big|_{t=0} = u_0 & \text{(initial condition)} \\ u\big|_{x=0} = 0, \quad u\big|_{x=L} = 0 & \text{(boundary conditions)} \end{cases}$$

Overview of the method:

- Decompose the heat equation into Fourier modes (in x)

- The PDE becomes an ODE (in time) for the Fourier coefficients: Solve it
- Find a formula for the solution by summing the mode solutions.

Remark (uniqueness) The initial value problem for the heat equation has a unique solution: If u_1, u_2 are two solutions, then $v = u_1 - u_2$ solves

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial^2 v}{\partial x^2} = 0 & \text{for } t>0,\ 0<x<L \\ v\big|_{t=0} = 0 & \\ v\big|_{x=0} = 0 = v\big|_{x=L} & \end{cases}$$

Then: $$\frac{d}{dt}\left(\int_0^L v^2(x,t)\,dx\right) = \int_0^L 2v\cdot\frac{\partial v}{\partial t}\,dx$$

$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}=0$

$$= 2\int_0^L v\cdot\frac{\partial^2 v}{\partial x^2}\,dx$$

Int. by parts

$$= 2\left(\left[v\cdot\frac{\partial v}{\partial x}\right]_{x=0}^{x=L} - \int_0^L\left(\frac{\partial v}{\partial x}\right)^2 dx\right)$$

$v|_{x=0}=0=v|_{x=L}$

$$= -2\int_0^L\left(\frac{\partial v}{\partial x}\right)^2 dx \le 0$$

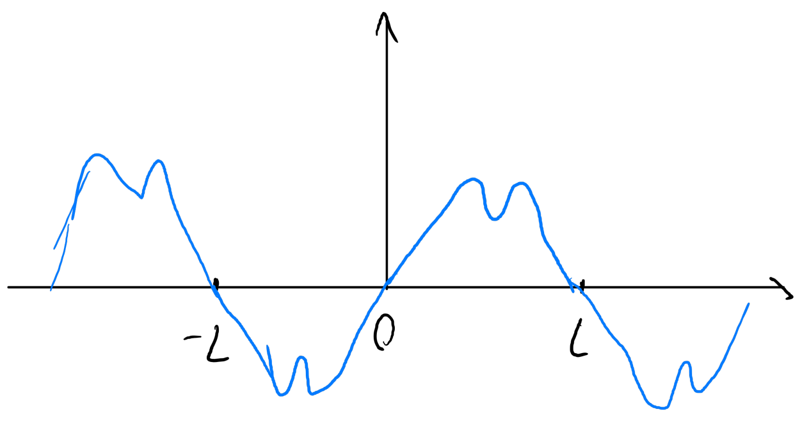
So $\int_0^L v^2(t,x)\,dx$ is decreasing

$$\Rightarrow \int_0^L v^2(t,x)\,dx \le \int_0^L v^2(0,x)\,dx \overset{v|_{t=0}}{=} 0$$

$$\Rightarrow V(t,x) = 0 \quad \forall\, t\ge 0,\ 0\le x\le L.$$

So, in view of uniqueness: If I can find a solution in any way, then that is the unique solution.

To decompose into Fourier modes: It helps to do the following trick (since I want $u|_{x=0} = 0$, $u|_{x=L} = 0$):



Extend $u(t,x)$ in x to a 2L-periodic function on $\mathbb{R}$ which is odd.

Do the same for u_0, f.

Note: If u is a $2L$-periodic and odd function which is continuous, then necessarily $u|_{x=0} = 0$ and $u|_{x=L} = 0$.

For the solution of the heat equation:
It can be shown that if f is piecewise continuous and the same is true for u_0: Then for $t>0$, $u(t,x)$ and $\frac{\partial u}{\partial x}(t,x)$ are continuous in x.

Fourier series (with respect to the 2L-period)

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \cdot \sin\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x\right)$$

$$b_n(t) = \frac{2}{L} \int_0^L u(x,t) \cdot \sin\left(\frac{\pi n}{L} x\right) dx$$

$$u_0(x) = \sum_{n=1}^{\infty} b_{0n} \cdot \sin\left(\frac{\pi n}{L} x\right)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \cdot \sin\left(\frac{\pi n}{L} x\right)$$

$$\frac{\partial u}{\partial x}(x,t) = \sum_{n=1}^{\infty} b_n(t) \cdot \frac{\pi n}{L} \cdot \cos\left(\frac{\pi n}{L} x\right)$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} b_n(t) \cdot \left(-\frac{\pi^2 n^2}{L^2}\right) \cdot \sin\left(\frac{\pi n}{L} x\right)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n'(t) \cdot \sin\left(\frac{\pi n}{L} x\right)$$

So $\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = f$ gives

$$\sum_{n=1}^{\infty} \left(b_n'(t) - \frac{\pi^2 n^2}{L^2} b_n(t) \right) \cdot \sin\left(\frac{\pi n}{L} x\right) = \sum_{n=1}^{\infty} f_n(t) \cdot \sin\left(\frac{\pi n}{L} x\right)$$

So $\boxed{b_n' - \dfrac{\pi^2 n^2}{L^2} b_n = f_n}$ for $n = 1, 2, 3, \dots$

for $t > 0$

Initial conditions $u(x,0) = u_0(x)$

So $\boxed{b_n(0) = b_{0n}}$

The above is an initial value problem of the form

$$\begin{cases} y' + ay = F \\ y(0) = y_0 \end{cases}$$

We have seen how to solve it (~2 weeks ago) using the Laplace transform.

Solution:

$$b_n(t) = b_{0n} \cdot e^{-\frac{\pi^2 n^2}{L^2} t} + \int_0^t e^{-\frac{\pi^2 n^2}{L^2}(t-s)} f_n(s)\, ds$$

So we have found u:

$$u(t,x) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{\pi n}{L} x\right)$$

$$= \sum_{n=1}^{\infty} \left(\cdots\cdots \right) \cdot \sin\left(\frac{\pi n}{L} x\right)$$

Remarks

- If $f = 0$: $u(x,t) = \sum_{n=1}^{\infty} b_{0n} \cdot e^{-\frac{\pi^2 n^2}{L^2} t} \cdot \sin\left(\frac{\pi n}{L} x\right)$

Decays exponentially fast to 0 as $t \longrightarrow +\infty$ (and faster decay in higher modes)

- Special solutions: If $u_0 = 0$, $f(x,t) = \sin\left(\frac{\pi x}{L}\right)$:

$$\Rightarrow \quad f_n(t) = \begin{cases} 1, & n = 1 \\ 0, & n \geq 2 \end{cases}$$

$$\Rightarrow \quad b_n(t) = \begin{cases} \frac{L^2}{\pi^2}\left(1 - e^{-\frac{\pi^2}{L^2}t}\right), & n = 1 \\ 0, & n \geq 2 \end{cases}$$

So $$u(x,t) = \frac{L^2}{\pi^2} \cdot \left(1 - e^{-\frac{\pi^2}{L^2}t}\right) \cdot \sin\left(\frac{\pi x}{L}\right).$$

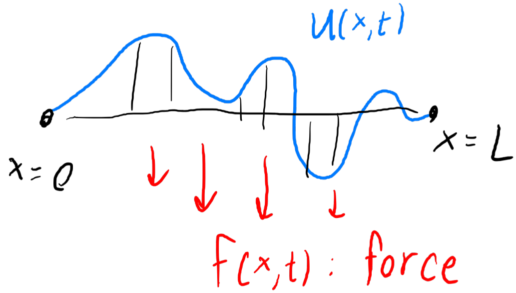
- If we wanted Neuman boundary conditions (namely $\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0 = \left.\frac{\partial u}{\partial x}\right|_{x=L}$)

We would extend u as an even 2L-periodic function. In that case:

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cdot \cos\left(\frac{\pi n x}{L}\right).$$

Wave equation:

Elastic cord:



$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = f & \text{for } t>0,\ 0<x<L \\ u\big|_{t=0} = u_0 & \text{(initial position)} \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = u_1 & \text{(initial velocity)} \\ u\big|_{x=0} = 0,\quad u\big|_{x=L} = 0 & \text{Dirichlet boundary condition: Fixed endpoints} \end{cases}$$

$c \neq 0$: Speed of propagation.

Remarks

- Energy conservation:

  If we define

$$E[u](t) = \frac{1}{2}\int_0^L \underbrace{\left(\frac{\partial u}{\partial t}\right)^2(x,t)}_{\text{Kinetic}} + \underbrace{c^2\left(\frac{\partial u}{\partial x}\right)^2(x,t)}_{\text{Potential energy}}\, dx,$$

then

$$\frac{d}{dt}E[u](t) = \frac{1}{2}\int_0^L 2\,\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial t^2} + 2c^2\,\frac{\partial u}{\partial x}\,\frac{\partial^2 u}{\partial x\partial t}\,dx$$

$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F$

$$= \int_0^L \frac{\partial u}{\partial t}\cdot\left(F + c^2\frac{\partial^2 u}{\partial x^2}\right) + c^2\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t}\,dx$$

Integrate by parts in $\frac{\partial u}{\partial t}\cdot\frac{\partial^2 u}{\partial x^2}$

$$= \int_0^L \frac{\partial u}{\partial t}\cdot f\,dx + \left[c^2\cdot\frac{\partial u}{\partial t}\cdot\frac{\partial u}{\partial x}\right]_{x=0}^{x=L}$$

$$- \int_0^L \cancel{c^2\cdot\frac{\partial^2 u}{\partial t\partial x}\cdot\frac{\partial u}{\partial x}\,dx} + \int_0^L \cancel{\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x\partial t}\,dx}$$

Since $u(0,t) = 0 = u(L,t)$

$\Longrightarrow \frac{\partial u}{\partial t}(0,t) = 0 = \frac{\partial u}{\partial t}(L,t)$

$$= \int_0^L \frac{\partial u}{\partial t} \cdot f \, dx$$

So $$\frac{d}{dt} E[u](t) = \int_0^L \frac{\partial u}{\partial t}(x,t) \cdot f(x,t) \, dx$$

- The conservation of energy implies the uniqueness of solutions: If u_1, u_2 are two solutions and $v = u_1 - u_2$, then v solves

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = 0 \\ v|_{t=0} = 0 \\ \frac{\partial v}{\partial t}\Big|_{t=0} = 0 \\ v|_{x=0} = 0, \quad v|_{x=L} = 0 \end{cases}$$

Conservation of energy for v:

$$\frac{d}{dt} E[v](t) = 0 \qquad (\text{since } f=0 \text{ for } v)$$

$$\Rightarrow \forall t \geq 0: \ E[v](t) = E[v](0).$$

But $v|_{t=0}$ and $\left.\frac{\partial v}{\partial t}\right|_{t=0}$ both vanish,

so $E[v](0) = 0$

$$\Rightarrow \forall t \geq 0: \ E[v] = 0.$$

Since $E[v](t) = \frac{1}{2} \int_0^L \left(\frac{\partial v}{\partial t}\right)^2 + c^2 \left(\frac{\partial v}{\partial x}\right)^2 dx$,

we infer that $\frac{\partial v}{\partial t} = 0$ and $\frac{\partial v}{\partial x} = 0 \ \forall t, x$

So $v = \text{const} \xrightarrow{v|_{t=0} = 0} v = 0 \iff u_1 = u_2.$